Completing the square with 2 variables

Deciding Which Method to Use when Solving Quadratic Equations When solving a quadratic equation, follow these steps (in this order) to decide on a method: Try first to solve the equation by factoring. Be sure that your equation is in standard form (ax2+bx+c=0) before you start your factoring attempt. Don't waste a lot of time trying to factor your equation; if you can't get it factored in less than 60 seconds, move on to another method. Next, look at the side of the equation containing the variable. Is that side a perfect square? If it is, then you can solve the equation by taking the square root of both sides of the equation. Don't forget to include a ± sign in your equation once you have taken the square root. Next, if the coefficient of the squared term is 1 and the coefficient of the linear (middle) term is even, completing the square is a good method to use. Finally, the quadratic formula will work on any quadratic equation. However, if using the formula results in awkwardly large numbers under the radical sign, another method of solving may be a better choice. Now we'll look at some equations and think about the most appropriate method for solving them.   Example 1: Solve x2 + 4 = 4x First, put the equation in standard form so that we can try to solve it by factoring: x2 - 4x + 4 = 0 (x - 2)(x - 2) = 0 x - 2 = 0 | x - 2 = 0 x = 2 | x = 2 So the solution to this equation, found by factoring, is x = 2.   Example 2: Solve (2x - 2)2 = -4 The side of the equation containing the variable (the left side) is a perfect square, so we'll take the square root of both sides to solve the equation. (2x - 2)2 = -4 2x - 2 = ± 2i 2x = 2 ± 2i x = 1 ± i Notice that the ± sign was inserted in the equation at the point that the square root was taken.   Example 3: Solve x2 + 6x - 11 = 0 This equation is not factorable, and the side containing the variable is not a perfect square. But since the coefficient of the x2 is 1 and the coefficient of the x is even, completing the square will be an appropriate method. To find the number which needs to be added to both sides of the equation to complete the square, take the coefficient of the x term, divide it by 2, then square that number. In this problem, 6 ¸ 2 is 3, and 32 is 9, so we'll add 9 to both sides of the equation once we have isolated the variable terms. x2 + 6x - 11 = 0 x2 + 6x = 11 x2 + 6x +9 = 11 + 9 (x + 3)2 = 20 Example 4: Solve 2x2 - x + 5 = 0 This equation is not factorable, the left side is not a perfect square, and the coefficients of the x2 and x terms will not make completing the square convenient. That leaves the quadratic formula as the best method for solving this equation. We'll use a=2, b=-1, and c=5. Learning Outcomes Complete the square to solve a quadratic equation. Use the quadratic formula to solve a quadratic equation. Use the discriminant to determine the number and type of solutions to a quadratic equation. Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use other methods for solving a quadratic equation. Completing the Square One method is known as completing the square. Using this process, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, [latex]a[/latex], must equal 1. If it does not, then divide the entire equation by [latex]a[/latex]. Then, we can use the following procedures to solve a quadratic equation by completing the square. We will use the example [latex]{x}^{2}+4x+1=0[/latex] to illustrate each step. Given a quadratic equation that cannot be factored and with [latex]a=1[/latex], first add or subtract the constant term to the right sign of the equal sign. [latex]{x}^{2}+4x=-1[/latex] Multiply the b term by [latex]\frac{1}{2}[/latex] and square it. [latex]\begin{array}{l}\frac{1}{2}\left(4\right)=2\hfill \\ {2}^{2}=4\hfill \end{array}[/latex] Add [latex]{\left(\frac{1}{2}b\right)}^{2}[/latex] to both sides of the equal sign and simplify the right side. We have [latex]\begin{array}{l}{x}^{2}+4x+4=-1+4\hfill \\ {x}^{2}+4x+4=3\hfill \end{array}[/latex] The left side of the equation can now be factored as a perfect square. [latex]\begin{array}{l}{x}^{2}+4x+4=3\hfill \\ {\left(x+2\right)}^{2}=3\hfill \end{array}[/latex] Use the square root property and solve. [latex]\begin{array}{l}\sqrt{{\left(x+2\right)}^{2}}=\pm \sqrt{3}\hfill \\ x+2=\pm \sqrt{3}\hfill \\ x=-2\pm \sqrt{3}\hfill \end{array}[/latex] The solutions are [latex]x=-2+\sqrt{3}[/latex], [latex]x=-2-\sqrt{3}[/latex]. Solve the quadratic equation by completing the square: [latex]{x}^{2}-3x - 5=0[/latex]. Solve by completing the square: [latex]{x}^{2}-6x=13[/latex]. Using the Quadratic Formula The fourth method of solving a quadratic equation is by using the quadratic formula, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number. We can derive the quadratic formula by completing the square. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by [latex]-1[/latex] and obtain a positive a. Given [latex]a{x}^{2}+bx+c=0[/latex], [latex]ae 0[/latex], we will complete the square as follows: First, move the constant term to the right side of the equal sign: [latex]a{x}^{2}+bx=-c[/latex] As we want the leading coefficient to equal 1, divide through by a: [latex]{x}^{2}+\frac{b}{a}x=-\frac{c}{a}[/latex] Then, find [latex]\frac{1}{2}[/latex] of the middle term, and add [latex]{\left(\frac{1}{2}\frac{b}{a}\right)}^{2}=\frac{{b}^{2}}{4{a}^{2}}[/latex] to both sides of the equal sign: [latex]{x}^{2}+\frac{b}{a}x+\frac{{b}^{2}}{4{a}^{2}}=\frac{{b}^{2}}{4{a}^{2}}-\frac{c}{a}[/latex] Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction: [latex]{\left(x+\frac{b}{2a}\right)}^{2}=\frac{{b}^{2}-4ac}{4{a}^{2}}[/latex] Now, use the square root property, which gives [latex]\begin{array}{l}x+\frac{b}{2a}=\pm \sqrt{\frac{{b}^{2}-4ac}{4{a}^{2}}}\hfill \\ x+\frac{b}{2a}=\frac{\pm \sqrt{{b}^{2}-4ac}}{2a}\hfill \end{array}[/latex] Finally, add [latex]-\frac{b}{2a}[/latex] to both sides of the equation and combine the terms on the right side. Thus, [latex]x=\frac{-b\pm \sqrt{{b}^{2}-4ac}}{2a}[/latex] Written in standard form, [latex]a{x}^{2}+bx+c=0[/latex], any quadratic equation can be solved using the quadratic formula: [latex]x=\frac{-b\pm \sqrt{{b}^{2}-4ac}}{2a}[/latex] where a, b, and c are real numbers and [latex]ae 0[/latex]. How To: Given a quadratic equation, solve it using the quadratic formula Make sure the equation is in standard form: [latex]a{x}^{2}+bx+c=0[/latex]. Make note of the values of the coefficients and constant term, [latex]a,b[/latex], and [latex]c[/latex]. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula. Calculate and solve. Solve the quadratic equation: [latex]{x}^{2}+5x+1=0[/latex]. Use the quadratic formula to solve [latex]{x}^{2}+x+2=0[/latex]. . Notice they are written in standard form of a complex number. When a solution is a complex number, you must separate the real part from the imaginary part and write it in standard form. Solve the quadratic equation using the quadratic formula: [latex]9{x}^{2}+3x - 2=0[/latex]. The quadratic formula not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the discriminant, or the expression under the radical, [latex]{b}^{2}-4ac[/latex]. The discriminant tells us whether the solutions are real numbers or complex numbers as well as how many solutions of each type to expect. The table below relates the value of the discriminant to the solutions of a quadratic equation. Value of Discriminant Results [latex]{b}^{2}-4ac=0[/latex] One rational solution (double solution) [latex]{b}^{2}-4ac>0[/latex], perfect square Two rational solutions [latex]{b}^{2}-4ac>0[/latex], not a perfect square Two irrational solutions [latex]{b}^{2}-4ac 0, may be expressed in terms of the square of the absolute value of a complex number. Define z = a x + i b y . {\displaystyle z={\sqrt {a}}\,x+i{\sqrt {b}}\,y.} Then | z | 2 = z z ∗ = (a x + i b y) (a x − i b y) = a x 2 − i a b x y + i b a y x − i 2 b y 2 = a x 2 + b y 2 , {\displaystyle {\begin{aligned}|z|^{2}&{}=zz^{*}\\&{}=({\sqrt {a}}\,x+i{\sqrt {b}}\,y)({\sqrt {a}}\,x-i{\sqrt {b}}\,y)\\&{}=ax^{2}-i{\sqrt {ab}}\,xy+i{\sqrt {ba}}\,yx-i^{2}by^{2}\\&{}=ax^{2}+by^{2},\end{aligned}}} so a x 2 + b y 2 + c = | z | 2 + c . {\displaystyle ax^{2}+by^{2}+c=|z|^{2}+c.} Idempotent matrix A matrix M is idempotent when M2 = M. Idempotent matrices generalize the idempotent properties of 0 and 1. The completion of the square method of addressing the equation a 2 + b 2 = a , {\displaystyle a^{2}+b^{2}=a,} shows that some idempotent 2 × 2 matrices are parametrized by a circle in the (a,b)-plane: The matrix (a b b 1 − a) {\displaystyle {\begin{pmatrix}a&b\\b&1-a\end{pmatrix}}} will be idempotent provided a 2 + b 2 = a , {\displaystyle a^{2}+b^{2}=a,} which, upon completing the square, becomes (a − 1 2) 2 + b 2 = 1 4 . {\displaystyle (a-{\tfrac {1}{2}})^{2}+b^{2}={\tfrac {1}{4}}.} In the (a,b)-plane, this is the equation of a circle with center (1/2, 0) and radius 1/2. Geometric perspective Consider completing the square for the equation x 2 + b x = a . {\displaystyle x^{2}+bx=a.} Since x2 represents the area of a square with side of length x, and bx represents the area of a rectangle with sides b and x, the process of completing the square can be viewed as visual manipulation of rectangles. Simple attempts to combine the x2 and the bx rectangles into a larger square result in a missing corner. The term (b/2)2 added to each side of the above equation is precisely the area of the missing corner, whence derives the terminology "completing the square". A variation on the technique As conventionally taught, completing the square consists of adding the third term, v 2 to u 2 + 2 u v {\displaystyle u^{2}+2uv} to get a square. There are also cases in which one can add the middle term, either 2uv or −2uv, to u 2 + v 2 {\displaystyle u^{2}+v^{2}} to get a square. Example: the sum of a positive number and its reciprocal By writing x + 1 x = (x − 2 + 1 x) + 2 = (x − 1 x) 2 + 2 {\displaystyle {\begin{aligned}x+{1 \over x}&{}=\left(x-2+{1 \over x}\right)+2\\&{}=\left({\sqrt {x}}-{1 \over {\sqrt {x}}}\right)^{2}+2\end{aligned}}} we show that the sum of a positive number x and its reciprocal is always greater than or equal to 2. The square of a real expression is always greater than or equal to zero, which gives the stated bound; and here we achieve 2 just when x is 1, causing the square to vanish. Example: factoring a simple quartic polynomial Consider the problem of factoring the polynomial x 4 + 324. {\displaystyle x^{4}+324.} This is (x 2) 2 + (18) 2 , {\displaystyle (x^{2})^{2}+(18)^{2},} so the middle term is 2(x2)(18) = 36x2. Thus we get x 4 + 324 = (x 4 + 36 x 2 + 324) − 36 x 2 = (x 2 + 18) 2 − (6 x) 2 = a difference of two squares = (x 2 + 18 + 6 x) (x 2 + 18 − 6 x) = (x 2 + 6 x + 18) (x 2 − 6 x + 18) {\displaystyle {\begin{aligned}x^{4}+324&{}=(x^{4}+36x^{2}+324)-36x^{2}\\&{}=(x^{2}+18)^{2}-(6x)^{2}={\text{a difference of two squares}}\\&{}=(x^{2}+18+6x)(x^{2}+18-6x)\\&{}=(x^{2}+6x+18)(x^{2}-6x+18)\end{aligned}}} (the last line being added merely to follow the convention of decreasing degrees of terms). The same argument shows that x 4 + 4 a 4 {\displaystyle x^{4}+4a^{4}} is always factorizable as x 4 + 4 a 4 = (x 2 + 2 a x + 2 a 2) (x 2 − 2 a x + 2 a 2) {\displaystyle x^{4}+4a^{4}=(x^{2}+2ax+2a^{2})(x^{2}-2ax+2a^{2})} (Also known as Sophie-Germain Identity). References ^ Narasimhan, Revathi (2008). Precalculus: Building Concepts and Connections. Cengage Learning. pp. 133–134. ISBN 0-618-41301-4., Section Formula for the Vertex of a Quadratic Function, page 133–134, figure 2.4.8 Algebra 1, Glencoe, ISBN 0-07-825083-8, pages 539–544 Algebra 2, Saxon, ISBN 0-939798-62-X, pages 214–214, 241–242, 256–257, 398–401 External links Wikimedia Commons has media related to Completing the square. Completing the square at PlanetMath. Retrieved from "